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FREE VIBRATIONS OF CURVED TIMOSHENKO BEAMS ON PASTERNAK FOUNDATIONS

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Abstract—A study of the natural vibration of a continuous Timoshenko curved beam on a Pasternak-type foundation is presented. The dynamic stiffness matrix of a curved member of constant section is derived. An example of a two-span curved beam is given to illustrate the application of the proposed method and to show the effects of flexural and torsional rotary inertia, shear deformation, central angle of the arc, contact area between the beam and foundation, and the foundation constants on the natural frequencies of the beam.

1. INTRODUCTION

The problem of beams on elastic foundations occupies an important place in modern structural and foundation engineering. The static case has been studied extensively, and the subject is covered in great depth by Volterra (1952, 1953) and Panayotounakas and Theocaris (1979). For the dynamic case, most works have been done within the scope of elementary Bernoulli–Euler beams on elastic foundations. Usually, the subgrade is replaced either by Winkler foundation, (Hetenyi, 1966) or by a homogeneous, isotropic semi-infinite elastic continuum (Richart *et al.*, 1970). However, Kerr (1964) has shown that there is a large class of foundation materials occurring in engineering practice the behavior of which cannot be represented by these two models. In an attempt to find a physically close and mathematically simple representation of an elastic foundation for these materials, Pasternak and Izdat (1954) proposed a foundation model consisting of a Winkler foundation with shear interaction. This may be accomplished by connecting the ends of the vertical springs to a beam consisting of the compressible vertical elements, which deforms only the transverse shear. Rades (1970) studied the steady-state response of a beam on a Pasternak-type foundation. His results have shown that, for the bending moments, the responses obtained when using the Pasternak foundation model are totally different from those for the Winkler model. The insufficiency of the Winkler model in his study is emphasized.

The Bernoulli–Euler theory of flexural vibrations of beams is adequate for relatively long, slender beams at lower modes of vibration. For beams having large cross-sectional dimensions in comparison to their lengths, and for beams in which higher modes are required, the Timoshenko theory gives a better approximation to the true behaviour of a beam (Huang, 1961; Issa, 1988). The application of the Timoshenko theory to beam vibration has been considered by Issa *et al.* (1987). They presented the frequency equations and normal modes of free vibrations for curved beams. Issa (1988) investigated the vibrations of continuous Timoshenko curved beams on a Winkler foundation. Loura and Gutierrez (1985) presented an analytical solution to the problem of vibrating nonuniform plates on an elastic foundation; free and forced vibration were studied.

In the present paper, the effects of Pasternak foundations on natural frequencies of finite Timoshenko curved beams are studied. The governing system of partial differential equations is presented first. The general solution of these equations is then obtained and the dynamic stiffness matrix is derived. Numerical results are given to show the effects of flexural and torsional rotary inertia, shear deformation, central angle of the arc, contact area between the beam and foundation, and the foundation constants on the natural frequencies of the beam. The mass of the foundation will, apparently, have an effect on natural frequencies. This effect has not been considered in the work presented here. However, a study of this effect is underway.

2. DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

The coupled differential equations for transverse vibrations of Timoshenko curved beams on Winkler foundations take the forms as given by Issa (1988):

$$\frac{KAGR^2}{EI}\frac{\partial v}{\partial \theta}+R\frac{\partial^2\phi}{\partial^2\theta}-\left(u+\frac{KAG}{EI}R^2\right)R\phi-\frac{\gamma R^3}{E}\frac{\partial^2\phi}{\partial t^2}-(1+u)R\frac{\partial\psi}{\partial\theta}=0 \quad (1)$$

$$(1+u)R\frac{\partial\phi}{\partial\theta}+uR\frac{\partial^2\psi}{\partial\theta^2}-\frac{\gamma R^3 I_p}{EI}\frac{\partial^2\psi}{\partial t^2}-R\psi-\frac{R^3}{EI}t_f=0 \quad (2)$$

$$\frac{KAGR^2}{EI}\frac{\partial^2 v}{\partial\theta^2}-\frac{\gamma AR^4}{EI}\frac{\partial^2 v}{\partial t^2}-\frac{KAGR^3}{EI}\frac{\partial\phi}{\partial\theta}-\frac{R^4}{EI}P_f=0 \quad (3)$$

in which E = modulus of elasticity, G = modulus of rigidity, J = torsional constant, K = numerical shape factor of the cross-section, I = moment of inertia of the cross-section, A = cross-sectional area, γ = mass per unit volume, I_P = polar moment of inertia of cross-sectional area, ϕ = bending slope of the cross-sectional area, v = vertical displacement of the center line of the curved beams, ψ = angle of twist of the beam cross-section, R = radius of curvature of the curved beam, u = stiffness factor, P_f = foundation vertical reaction per unit length, t_f = foundation torsional reaction and t = time.

The general form for a Pasternak foundation in the radial coordinate case has been derived as:

$$P_f=K_0cv-G_0\left[\frac{4c}{4R^2-c^2}\frac{\partial^2 v}{\partial\theta^2}-\psi\ln\frac{2R+c}{2R-c}+\left(\frac{4cR}{4R^2-c^2}-\ln\frac{2R+c}{2R-c}\right)\frac{\partial^2\psi}{\partial\theta^2}\right] \quad (4)$$

$$t_f=K_0(c^3/12)\psi+G_0\left[\left(\ln\frac{2R+c}{2R-c}-\frac{4cR}{4R^2-c^2}\right)\frac{\partial^2 v}{\partial\theta^2}+\left(R\ln\frac{2R+c}{2R-c}-c\right)\psi\right.$$
$$\left.-\left(c+\frac{4cR^2}{4R^2-c^2}-2R\ln\frac{2R+c}{2R-c}\right)\frac{\partial^2\psi}{\partial\theta^2}\right] \quad (5)$$

where k_0 = Winkler foundation modulus, G_0 = shear foundation modulus and c = width of the contact area between the beam and the foundation. With P_f and t_f expressed by eqns (4) and (5), eqns (2) and (3) become

$$\frac{G_0R^3}{EI}\left(\frac{4cR}{4R^2-c^2}-\ln\frac{2R+C}{2R-c}\right)\frac{\partial^2 v}{\partial\theta^2}+(1+u)R\frac{\partial\phi}{\partial\theta}$$
$$+\left[uR+\frac{G_0R^3}{EI}\left(c+\frac{4cR^2}{4R^2-c^2}-2R\ln\frac{2R+c}{2R-c}\right)\right]\frac{\partial^2\psi}{\partial\theta^2}-\frac{\gamma I_pR^3}{EI}\frac{\partial^2\psi}{\partial t^2}$$
$$-\left[1+\frac{K_0c^3R^2}{12EI}+\frac{G_0R^2}{EI}\left(R\ln\frac{2R+c}{2R-c}-c\right)\right]R\psi=0 \quad (6)$$

$$\left(\frac{KAGR^2}{EI}+\frac{4cG_0R^4}{EI(4R^2-c^2)}\right)\frac{\partial^2 v}{\partial\theta^2}-\frac{cK_0R^2}{EI}v-\frac{\gamma AR^4}{EI}\frac{\partial^2 v}{\partial t^2}-\frac{KAGR^3}{EI}\frac{\partial\phi}{\partial\theta}$$
$$+\frac{G_0R^3}{EI}\left(\frac{4cR^2}{4R^2-c^2}-R\ln\frac{2R+c}{2R-c}\right)\frac{\partial^2\psi}{\partial\theta^2}-\frac{G_0R^4}{EI}\left(\ln\frac{2R+c}{2R-c}\right)\psi=0. \quad (7)$$

Let

$$v(\theta, t) = V(\Theta)\, e^{i\omega t} \tag{8}$$

$$\phi(\theta, t) = \Phi(\Theta)\, e^{i\omega t} \tag{9}$$

$$\psi(\theta, t) = \Psi(\Theta)\, e^{i\omega t} \tag{10}$$

where $i = \sqrt{-1}$, ω = angular frequency, V = normal function of v, Φ = normal function of ϕ, Ψ = normal function of ψ. When the common factor $e^{i\omega t}$ is omitted, eqns (1) and (6)–(7) reduce to

$$\frac{1}{s^2\alpha^2}V' + L\Phi'' + \left(\frac{b^2r^2}{\alpha^2} - \frac{1}{s^2\alpha^2} - u\right)L\Phi - (1+u)L\Psi' = 0 \tag{11}$$

$$\frac{z^2}{\alpha^2}(c_2 - c_3)V'' + (1+u)L\Phi' + \left[u + \frac{z^2}{\alpha^3}(c_1 + c_2 - 2c_3)\right]L\Psi'' + \left[\frac{b^2\alpha^2}{\alpha^2} - \frac{w^2}{\alpha^5}c_4 + \frac{z^2}{\alpha^3}(c_1 - c_3) - 1\right]L\Psi = 0 \tag{12}$$

$$\left(\frac{1}{s^2\alpha^2} + \frac{z^2}{\alpha^2}c_2\right)V'' + \left(\frac{b^2}{\alpha^3} - \frac{w^2}{\alpha^4}c_1\right)V - \frac{1}{s^2\alpha^2}L\Phi' + \frac{z^2}{\alpha^3}(c_2 - c_3)L\Psi'' - \frac{z^2}{\alpha^3}c_3 L\Psi = 0 \tag{13}$$

where

$$\begin{aligned} &b^2 = \gamma A L^2\omega^2/(EI), \quad r^2 = I/(AL^2), \quad a^2 = I_P/(AL^2) \\ &s^2 = EI/(KAGL^2), \quad u = GJ/EI, \quad w^2 = K_0L^5/EI \\ &z^2 = G_0L^3/EI, \quad c_1 = c/R, \quad c_2 = 4cR/(4R^2 - c^2), \\ &c_3 = \ln[(2R+c)/(2R-c)], \quad c_4 = c^3/12R^3. \end{aligned} \tag{14}$$

By manipulation, eqns (11)–(13) can be decoupled and the following linear differential equations for V, Φ and Ψ result:

$$V^{VI} + A_1V^{IV} + A_2V'' + A_3V = 0 \tag{15}$$

$$L\Phi^{VI} + A_1L\Phi^{IV} + A_2L\Phi'' + A_3L\Phi = 0 \tag{16}$$

$$L\Psi^{VI} + A_1L\Psi^{IV} + A_2L\Psi'' + A_3L\Psi = 0. \tag{17}$$

The primes for V, Φ and Ψ represent differentiation with respect to θ. The coefficients A_1, A_2 and A_3 are as given in the Appendix.

The solution of eqns (15)–(17) may be expressed as:

$$V(\theta) = \sum_{n=1}^{6} C_n\, e^{\lambda_n\theta} \tag{18}$$

$$L\Phi(\theta) = \sum_{n=1}^{6} D_n\, e^{\lambda_n\theta} \tag{19}$$

$$L\Psi(\theta) = \sum_{n=1}^{6} E_n\, e^{\lambda_n\theta} \tag{20}$$

where C_n, D_n and E_n are constants to be determined from the boundary conditions, and λ_n ($n = 1, 2, \ldots, 6$) are the roots of the characteristic equation

$$\lambda^6 + A_1\lambda^4 + A_2\lambda^2 + A_3 = 0. \tag{21}$$

The relation between D_n, E_n and C_n can be obtained from eqns (11) and (12). These relationships can be written as

$$D_n = H_n C_n \tag{22}$$

$$E_n = G_n C_n \tag{23}$$

where

$$H_n = -\frac{(h_3 h_4 + h_1 h_5)\lambda_n^3 + h_1 h_6 \lambda_n}{h_5 \lambda_n^4 + (h_3^2 + h_2 h_5 + h_6)\lambda_n^2 + h_2 h_6} \tag{24}$$

$$G_n = -\frac{h_4 \lambda_n^4 + (h_2 h_4 - h_1 h_3)\lambda_n^2}{h_5 \lambda_n^4 + (h_3^2 + h_2 h_5 + h_6)\lambda_n^2 + h_2 h_6} \tag{25}$$

in which

$$h_1 = \frac{1}{s^2\alpha}, \quad h_2 = \frac{b^2 r^2}{\alpha^2} - \frac{1}{s^2\alpha^2} - u, \quad h_3 = 1 + u, \quad h_4 = \frac{z^2}{\alpha^2}(c_2 - c_3)$$

$$h_5 = u + \frac{z^2}{\alpha^3}(c_1 + c_2 - 2c_3), \quad h_6 = \frac{b^2 a^2}{\alpha^2} - \frac{w^2}{\alpha^5}c_4 + \frac{z^2}{\alpha^3}(c_1 - c_3) - 1. \tag{26}$$

Thus, eqns (19) and (20) become

$$L\Phi(\theta) = \sum_{n=1}^{6} H_n C_n \, e^{\lambda_n \theta} \tag{27}$$

$$L\Psi(\theta) = \sum_{n=1}^{6} G_n C_n \, e^{\lambda_n \theta}. \tag{28}$$

3. DYNAMIC STIFFNESS MATRIX

Figure 1 shows a horizontally circular curved member of constant cross-section, resting on Pasternak foundations, and subjected to harmonic displacements, linear and rotational, at the two ends a and b.

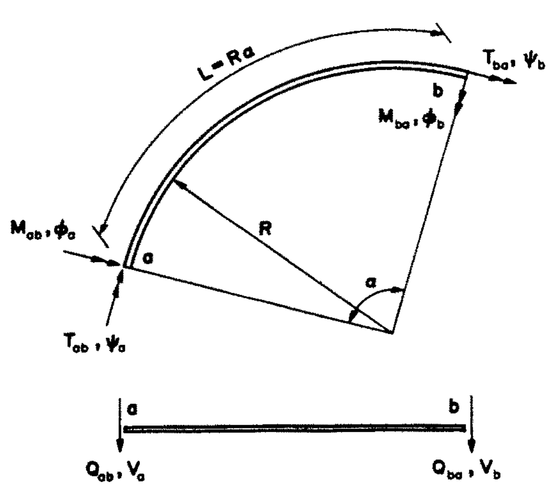

Fig. 1. Positive end displacements and forces of a circular curved member on Pasternak foundations.

The end displacements and forces of the curved member take the form as implemented by Issa (1988):

$$\{\delta\} = [A_0]\{X\}, \quad \{F\} = \frac{EI}{L^2}[B_0]\{X\} \tag{29,30}$$

where

$$\{\delta\} = \begin{Bmatrix} \phi_a \\ \Psi_a \\ V_a \\ \phi_b \\ \Psi_b \\ V_b \end{Bmatrix}, \quad \{F\} = \begin{Bmatrix} M_{ab} \\ T_{ab} \\ Q_{ab} \\ M_{ba} \\ T_{ba} \\ Q_{ba} \end{Bmatrix}, \quad \{X\} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix}$$

and matrices $[A_0]$ and $[B_0]$ are given as follows:

$$[A_0] = \begin{bmatrix} \frac{-H_1}{L} & \frac{-H_2}{L} & \frac{-H_3}{L} & \frac{-H_4}{L} & \frac{-H_5}{L} & \frac{-H_6}{L} \\ \frac{G_1}{L} & \frac{G_2}{L} & \frac{G_3}{L} & \frac{G_4}{L} & \frac{G_5}{L} & \frac{G_6}{L} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{-H_1}{L}e^{\lambda_1\alpha} & \frac{-H_2}{L}e^{\lambda_2\alpha} & \frac{-H_3}{L}e^{\lambda_3\alpha} & \frac{-H_4}{L}e^{\lambda_4\alpha} & \frac{-H_5}{L}e^{\lambda_5\alpha} & \frac{-H_6}{L}e^{\lambda_6\alpha} \\ \frac{G_1}{L}e^{\lambda_1\alpha} & \frac{G_2}{L}e^{\lambda_2\alpha} & \frac{G_3}{L}e^{\lambda_3\alpha} & \frac{G_4}{L}e^{\lambda_4\alpha} & \frac{G_5}{L}e^{\lambda_5\alpha} & \frac{G_6}{L}e^{\lambda_6\alpha} \\ e^{\lambda_1\alpha} & e^{\lambda_2\alpha} & e^{\lambda_3\alpha} & e^{\lambda_4\alpha} & e^{\lambda_5\alpha} & e^{\lambda_6\alpha} \end{bmatrix}$$

$$[B_0] = \begin{bmatrix} -m_1 & -m_2 & -m_3 & -m_4 & -m_5 & -m_6 \\ -t_1 & -t_2 & -t_3 & -t_4 & -t_5 & -t_6 \\ \frac{-q_1}{L} & \frac{-q_2}{L} & \frac{-q_3}{L} & \frac{-q_4}{L} & \frac{-q_5}{L} & \frac{-q_6}{L} \\ m_1 e^{\lambda_1\alpha} & m_2 e^{\lambda_2\alpha} & m_3 e^{\lambda_3\alpha} & m_4 e^{\lambda_4\alpha} & m_5 e^{\lambda_5\alpha} & m_6 e^{\lambda_6\alpha} \\ t_1 e^{\lambda_1\alpha} & t_2 e^{\lambda_2\alpha} & t_3 e^{\lambda_3\alpha} & t_4 e^{\lambda_4\alpha} & t_5 e^{\lambda_5\alpha} & t_6 e^{\lambda_6\alpha} \\ \frac{q_1}{L}e^{\lambda_1\alpha} & \frac{q_2}{L}e^{\lambda_2\alpha} & \frac{q_3}{L}e^{\lambda_3\alpha} & \frac{q_4}{L}e^{\lambda_4\alpha} & \frac{q_5}{L}e^{\lambda_5\alpha} & \frac{q_6}{L}e^{\lambda_6\alpha} \end{bmatrix}$$

where m_n, t_n and q_n are given in Issa (1988).

Eliminating $[X]$ from eqns (29) and (30) yields

$$\{F\} = [S_0]\{\delta\} \tag{31}$$

where $[S_0]$, the dynamic stiffness matrix for a horizontally circular curved member resting on Pasternak foundations, is given by

$$[S_0] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} = \frac{EI}{L^2}[B_0][A_0]^{-1}. \quad (32)$$

4. NUMERICAL EXAMPLE

A symmetrical circular curved beam resting on Pasternak-type foundations and having rigid nontwisting supports as shown in Fig. 2 is analyzed for natural frequencies.

The boundary conditions are

$$\Psi_A = \Psi_B = \Psi_C = 0, \quad V_A = V_B = V_C = 0, \quad (33)$$

and the conditions of dynamic equilibrium at A, B and C give

$$M_{AB} = 0, \quad M_{BA} + M_{BC} = 0, \quad M_{CB} = 0. \quad (34)$$

Since the beam has two identical spans,

$$[A_0]_{AB} = [A_0]_{BC} = [A_0], \quad [B_0]_{AB} = [B_0]_{BC} = [B_0], \quad [S_0]_{AB} = [S_0]_{BC} = [S_0]. \quad (35)$$

Thus eqns (31), (32) and (33) give

$$M_{AB} = S_{11}\Phi_A + S_{14}\Phi_B, \quad M_{BA} = S_{41}\Phi_A + S_{44}\Phi_B,$$
$$M_{BC} = S_{11}\Phi_B + S_{14}\Phi_C, \quad M_{CB} = S_{41}\Phi_B + S_{44}\Phi_C. \quad (36)$$

Substituting eqns (36) into eqns (34) yields a system of simultaneous equations in the following matrix form:

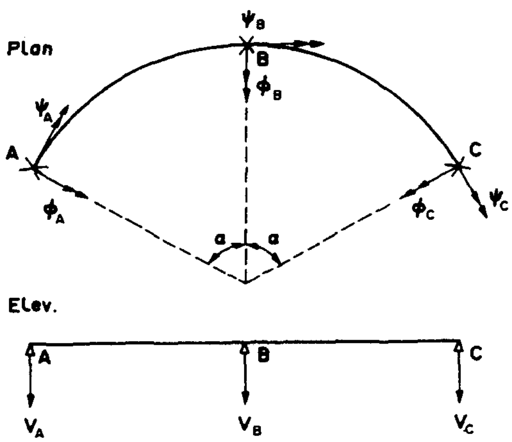

Fig. 2. A two-span circular curved beam on Pasternak foundations.

$$\begin{Bmatrix} M_{AB} \\ M_{EA}+M_{BC} \\ M_{CB} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{14} & 0 \\ S_{41} & S_{44}+S_{11} & S_{14} \\ 0 & S_{41} & S_{44} \end{bmatrix} \begin{Bmatrix} \Phi_A \\ \Phi_B \\ \Phi_C \end{Bmatrix} = \begin{matrix} 0 \\ 0 \\ 0. \end{matrix} \quad (37)$$

Equating the determinant of the stiffness matrix in eqn (37) to zero gives the frequency equation

$$\begin{bmatrix} S_{11} & S_{14} & 0 \\ S_{41} & S_{44}+S_{11} & S_{14} \\ 0 & S_{41} & S_{44} \end{bmatrix} = 0. \quad (38)$$

For a given curved beam, with α, L, r, s, u, c and w known, the frequency characteristics can be found from eqn (38). In order to show the effects of central angle, shear deformation, the foundation constants and rotary inertia with respect to flexure and torsion on the natural frequencies of the beam, the value of K is assumed to be 0.85 for rectangular beams for which: depth $= t$, breadth $= b$, $t = 2b$, Poisson's ratio $= 0.2$, $E = 200\ t\ \text{cm}^{-2}$ and $G = 83.4\ t\ \text{cm}^{-2}$. For this section the value of J is given by Huang (1961) as $J = 0.0286\ t^4$. Thus $u = GJ/EI = 0.286$, $a = 1.118\ r$, $s = 1.685\ r$. The ratio of the width of the contact area between the beam and the foundation, c, to the length of the beam, L, is taken as $C_1 = c/L = 0.08$; the foundation constants are taken as $w = 20$, $z = 10$.

Consider $r = 0.04$; then the values of b for $\alpha = 0°$ and $\alpha = 40°$ for the first five modes, obtained from eqn (38), are respectively

$$b_1 = 11.11, 15.49, 35.98, 42.89, 72.56,$$

$$b = 10.08, 14.67, 34.69, 41.81, 71.18.$$

Let ω_1 be the frequencies of a straight beam ($\alpha = 0$). Since $b/b_1 = \omega/\omega_1$, it follows that $\omega/\omega_1 = 0.907, 0.947, 0.964, 0.975, 0.981$.

The results of ω/ω_1 versus α for $r = 0.04$ for the first five modes are shown in Fig. 3. Curves given in Fig. 4 show the effects of shear deformation and rotary inertia with respect to flexure and torsion on the natural frequencies. Curves given in Fig. 5 show the effect of Pasternak shear modulus, G_0, on the natural frequencies of curved beams.

Curves given in Fig. 6 show the effect of C_1 on the natural frequencies of curved beams.

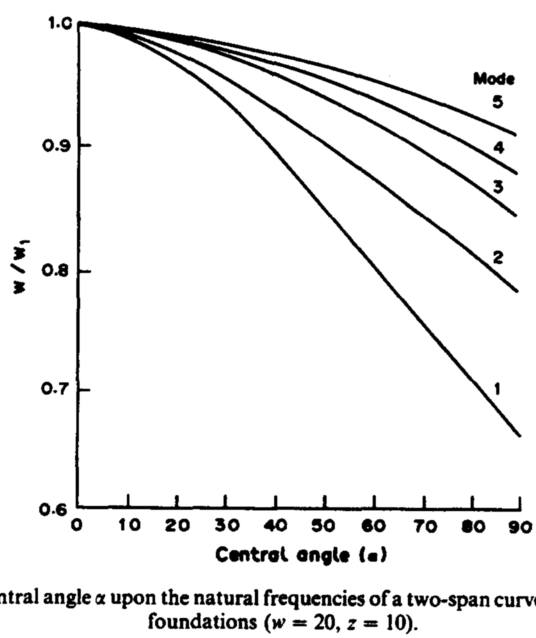

Fig. 3. Effect of central angle α upon the natural frequencies of a two-span curved beam on Pasternak foundations ($w = 20$, $z = 10$).

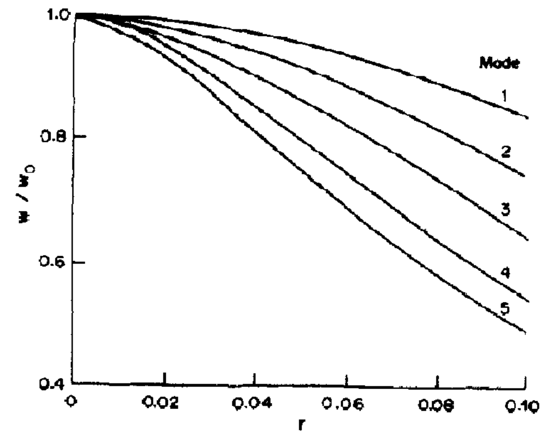

Fig. 4. Corrections in natural frequencies of a two-span curved beam on Pasternak foundations owing to shear deformation and rotary inertia with respect to flexure and torsion ($\alpha = 60°$, $w = 20$, $z = 10$).

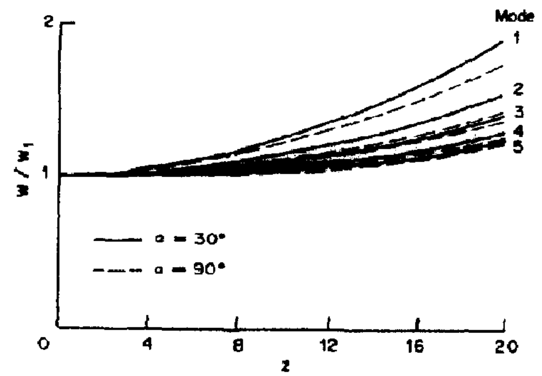

Fig. 5. Effect of Pasternak shear modulus, G_0, on the natural frequencies ($w = 20$).

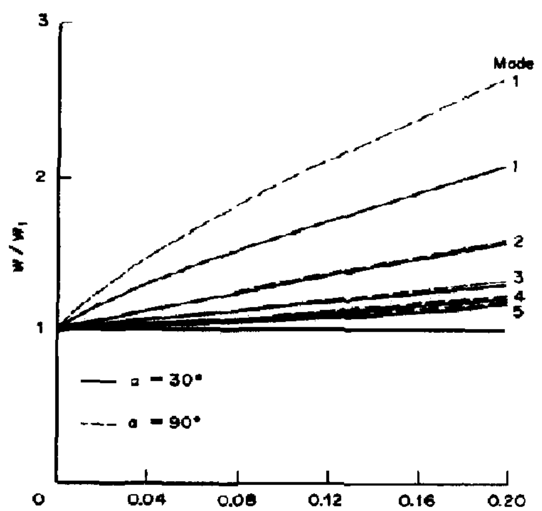

Fig. 6. Effect of width of the contact area between the beam and the foundations on the natural frequencies of two-span curved beam on Pasternak foundations ($w = 20$, $z = 10$).

5. CONCLUSIONS

The dynamic stiffness matrix formulation for circular curved members of constant cross-section resting on Pasternak-type foundations, including the effects of flexural and torsional rotary inertia and shear deformation, has been presented for the determination of the natural frequencies of continuous curved beams. The application of the proposed method has been illustrated in the example of a two-span curved beam resting on Pasternak foundations and undergoing out-of-plane vibrations. Among the most important results of this investigation one may list the following:

(1) The natural frequencies decrease as the central angle of the arc increases, and this effect becomes significant for lower modes. This is explained by the fact that the beam becomes more flexible as the central angle of the arc increases.

(2) The natural frequencies of curved beams increase as the Pasternak shear modulus increases. This effect is more pronounced for the lower modes and smaller values of the central angle.

(3) The natural frequencies of curved beams on Pasternak foundations are higher than those of curved beams on Winkler foundations. This reveals the effect of shear interactions in the Pasternak model.

(4) The natural frequencies increase as the width of the contact area between the beam and the foundation increases. This effect becomes more significant for the lower modes and for larger values of the central angle.

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APPENDIX

$$A_1 = (A_{CB}+A_W+A_P)/(1+x)$$

$$A_2 = (B_{CB}+B_W+B_P)/(1+x)$$

$$A_3 = (C_{CB}+C_W+C_P)/(1+x)$$

where

$$A_{CB} = 2+\frac{b^2}{\alpha^2}\left(r^2+\frac{a_2}{u}+\frac{1}{s^2}\right)$$

$$B_{CB} = 1+\frac{b^2}{\alpha^2}\left(2s^2-\frac{r^2}{u}-a^2-\frac{1}{\alpha^2}\right)+\frac{b^4}{\alpha^4}\left(\frac{r^2a^2}{u}+r^2s^2+\frac{a^2s^2}{u}\right)$$

$$C_{CB} = \frac{b^2}{\alpha^2}\left(s^2+\frac{1}{u\alpha^2}\right)-\frac{b^4}{\alpha^2}\left(\frac{r^2s^2}{u}+a^2s^2+\frac{a^2}{u\alpha^2}\right)+\frac{b^6}{\alpha^6}\frac{r^2a^2s^2}{u}$$

$$A_W = -\frac{w^2}{\alpha^2}\left(\frac{c_4}{\alpha^3 u}+\frac{c_1 s^2}{\alpha}\right)$$

$$B_W = -\frac{w^2}{\alpha^2}\left[\frac{(c_1+c_4)}{\alpha^3}-\frac{2c_1s^2}{\alpha}-\frac{b^2}{\alpha^2}\left(\frac{c_4r^2}{u\alpha^3}+\frac{c_1a^2s^2}{u\alpha}+\frac{c_1r^2s^2}{\alpha}+\frac{c_4s^2}{u\alpha^3}\right)\right]+\frac{w^4}{\alpha^4}\frac{c_1c_4s^2}{u\alpha^4}$$

$$C_W = -\frac{w^2}{\alpha^2}\left[\frac{c_1 s^2}{\alpha} + \frac{c_1}{u\alpha^3} - \frac{b^2}{\alpha^2}\left(\frac{c_1 r^2 s^2}{u\alpha} + \frac{c_1 a^2 s^2}{\alpha} + \frac{c_1 a^2}{u\alpha^3} + \frac{c_4 s^2}{\alpha^3} + \frac{c_4}{u\alpha^5}\right)\right.$$
$$\left. + \frac{b^4}{\alpha^4}\left(\frac{c_1 r^2 a^2 s^2}{u\alpha} + \frac{c_4 r^2 s^2}{u\alpha^3}\right)\right] - \frac{w^4}{\alpha^4}\left[c_1 c_4\left(\frac{s^2}{\alpha^4} + \frac{1}{u\alpha^6}\right) - \frac{b^2}{\alpha^2}\frac{c_1 c_4 r^2 s^2}{u\alpha^4}\right]$$

$$X = \frac{z^2}{\alpha^2}\left(\frac{c_1 + c_2 - 2c_3}{u\alpha} + c_2 s^2 \alpha\right) + \frac{z^4}{\alpha^4}\frac{(c_1 c_2 - c_3^2)s^2}{u}$$

$$A_P = \frac{z^2}{\alpha^2}\left\{\frac{c_3 - c_1}{u} - \frac{c_1}{\alpha} + \frac{2(c_2 - c_3)}{u\alpha} + 2c_2 \alpha s^2 - \frac{w^2}{\alpha^2}(c_1^2 + c_1 c_2 + c_2 c_4 - 2c_1 c_3)\frac{s^2}{u\alpha^2}\right.$$
$$\left. + \frac{b^2}{\alpha^2}\left[c_2 s^2 \alpha\left(\frac{a^2}{u} + r^2\right) + (c_1 + c_2 - 2c_3)\left(\frac{r^2}{u} + \frac{s^2}{u\alpha}\right)\right]\right\}$$
$$+ \frac{z^4}{\alpha^4}\left[(c_3^2 - c_1 c_2)\left(\frac{s^2}{u} + s^2 + \frac{1}{u\alpha^2}\right) + \frac{b^2}{\alpha^2}(c_1 c_2 - c_3^2)\frac{r^2 s^2}{u}\right]$$

$$B_P = \frac{z^2}{\alpha^2}\left\{\frac{(c_2 - c_3 - uc_1)}{u\alpha} + c_2 \alpha s^2 + \frac{w^2}{\alpha^2}\left[\frac{(c_1 c_3 - c_1^2)s^2}{u\alpha^2} + (c_1^2 + c_1 c_2 + c_2 c_4 - 2c_1 c_3)\left(\frac{s^2}{\alpha^2} + \frac{1}{u\alpha^4}\right)\right]\right.$$
$$+ \frac{b^2}{\alpha^2}\left[(c_1 - c_3)\left(\frac{s^2}{u\alpha} + \frac{r^2}{u\alpha}\right) - c_2\left(\frac{r^2 s^2 \alpha}{u} + a^2 s^2 \alpha + \frac{a^2}{u\alpha}\right) - (c_1 + c_2 - 2c_3)\left(\frac{s^2}{\alpha} + \frac{1}{u\alpha^3}\right)\right.$$
$$\left.\left. + \frac{w^2}{\alpha^2}(c_1^2 + c_1 c_2 + c_2 c_4 - 2c_1 c_3)\frac{r^2 s^2}{u\alpha^2}\right] + \frac{b^2}{\alpha^4}\left[\frac{c_2 r^2 a^2 s^2 \alpha}{u} + \frac{(c_1 + c_2 - 2c_3)r^2 s^2}{u\alpha}\right]\right\}$$
$$+ \frac{z^4}{\alpha^4}\left\{(c_3^2 - c_1 c_2)\left(s^2 + \frac{1}{u\alpha^2}\right) + \frac{b^2}{\alpha^2}(c_1 c_2 - c_3^2)\frac{r^2 s^2}{u}\right\}$$

$$C_P = \frac{z^2}{\alpha^2}\left\{\frac{w^2}{\alpha^2}(c_1^2 - c_1 c_3)\left(\frac{s^2}{\alpha^2} + \frac{1}{u\alpha^4}\right) + \frac{b^2}{\alpha^2}\left[\frac{(c_3 - c_1)s^2}{\alpha} + \frac{(c_3 - c_1)}{u\alpha^3} + \frac{w^2}{\alpha^2}(c_1 c_3 - c_1^2)\frac{r^2 s^2}{u\alpha^2}\right] + \frac{b^4}{\alpha^4}(c_1 - c_3)\frac{r^2 s^2}{u\alpha}\right\}.$$